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Scattering of Poincare waves by an irregular coastline. Part 2. Multiple scattering

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Kelvin and Poincaré waves are generated when an ocean wave arrives at a nominally rectilinear coastline and interacts with coastal irregularities. The discussion of this problem given by Howe & Mysak (1973) is extended in this paper in order to examine the role of multiple scattering of the Kelvin and Poincaré waves. An integro-differential kinetic equation is derived to describe these processes in the limit in which the irregularities are small compared with the characteristic wavelength. In the absence of dissipative mechanisms it is verified that this description of interactions with the coast conserves total wave energy. The theory is applied to a variety of idealized problems which model tidal and storm surge events, including the generation and decay of Kelvin waves by extensive and localized Poincaré-wave forcing, and the influence of multiple scattering on the radiation of Poincaré-wave noise into the open ocean.

1. Introduction

Howe & Mysak (1973, hereafter referred to as HM) have considered the reflexion and scattering of a Poincaré wave (i.e. a long gravity wave on a uniformly rotating sheet of homogeneous fluid, also known as a Sverdrup wave; see Platzman 1971) incident on a randomly irregular coastline. The coast was infinitely long and straight except for small deviations $\xi(y)$ which were assumed to be adequately represented by a centred, stationary random function of position y along the coast. A theory of energy transfer processes in random media was used to determine the total energy flux from the incident Poincaré (or P) wave into (i) a Kelvin (K) wave trapped against the coast and (ii) diffusely scattered P-wave noise. In HM the so-called binary scattering theory (Frisch 1968; Howe 1971) was applied and the consequences of multiple scattering were not taken into account. Multiple scattering is likely to have a significant effect in situations involving the propagation of a K-wave, say, over distances along the coast of the order of several characteristic wavelengths, or when a P-wave is incident or scattered at a grazing angle to the coastline. Such problems can be handled theoretically by extending a procedure which was used by Howe (1974*a*) to examine the multiple scattering of sound at an irregular surface and which makes use of a kinetic (or transport) equation for the wave energy.

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In § 2 of this paper we briefly recall certain results obtained in HM concerning the specular (or coherent) component of the field scattered from an incident P-wave. An integral equation for the distribution of the total scattered energy over all available wave modes which incorporates the effects of multiple interactions with the irregular coastline is then derived (§ 3) for the case in which the incident P-wave is harmonic in time. The various terms appearing in this equation are interpreted in § 4. In § 5 a modification is introduced to include the possibility of a slow space–time modulation of the scattered field; the result is an integro-differential equation of the type occurring in kinetic theory. This equation splits naturally into separate but coupled equations for the scattered K-wave and the diffusely distributed P-wave noise.

These equations are used (§§ 7–9) to study various idealized model problems which characterize simple storm surges and tidal events that occur in the neighbourhood of extensive irregular coastlines and in which K- and P-waves play a fundamental role. The solution of such problems must lead to an improved understanding of the relative importance of the various physical mechanisms involved, although no attempt is made in this paper to explain or interpret practical observations. In § 7 we examine the decay of a K-wave and determine the space–time behaviour of the K-wave generated by extensive (§ 8) and localized (§ 9) incident P-waves. Finally the P-wave transport equation is used in § 10 to assess the importance of multiple scattering of the diffusely scattered P-waves.

2. The specularly reflected field

Consider a plane harmonic P-wave for which the elevation of the free surface is given by

$$\phi_0 = a \exp i\{-l_0 x + m_0 y - \omega t\} + \text{c.c. (complex conjugate)} \tag{2.1}$$

and which is incident on the irregular coastline $x = \xi(y)$ from the half-space $x > \xi(y)$ (see figure 1). We suppose that $\xi(y)$ is a stationary random function of position y along a nominally rectilinear coast (the y axis) and has zero mean. Assume for definiteness that $\omega > 0$, in which case l_0 is also positive. The usual long-wave equations for a uniformly rotating sheet of fluid of undisturbed depth d (Lamb 1932, p. 319) imply that l_0, m_0 and ω are related by the following dispersion relation:

$$l_0^2 + m_0^2 \equiv K_0^2 = (\omega^2 - f^2)/s^2, \tag{2.2}$$

where $s^2 = gd$, g is the acceleration due to gravity and $f = 2\Omega \sin\theta_0$ is the Coriolis parameter at a mean latitude θ_0, Ω being the angular velocity of the earth. Note that (2.2) also represents the dispersion relation for an internal P-wave of vertical mode j if s^2 is replaced by $s_j^2 = gd_j$, where d_j is the 'equivalent depth', a quantity which depends on the density stratification. For most oceanic situations $d \gg d_1 > d_2 > \dots$. Also, for any given mode j, ϕ_0 is proportional to the internal wave amplitude at a given depth in the fluid.

The specularly reflected field ϕ_R is defined as the mean scattered wave with respect to an ensemble of realizations of the irregularities of the coastline and has the form

$$\phi_R = R \exp i\{l_0 x + m_0 y - \omega t\} + \text{c.c.}, \tag{2.3}$$

where [cf. HM (3.12)]

$$R = \frac{a[\omega l_0 - i m_0 f]}{\omega l_0 + i m_0 f}\{1 + O(K_0^2 \langle \xi^2 \rangle)\}, \tag{2.4}$$

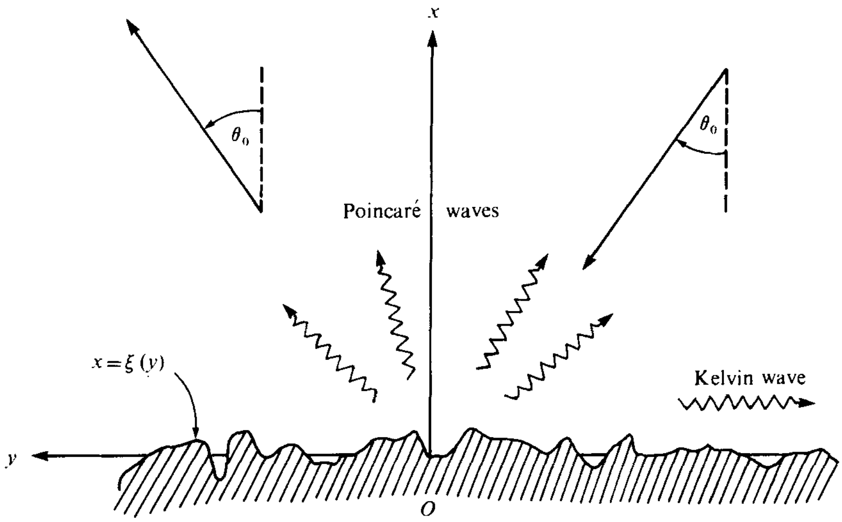

FIGURE 1. Irregular coastline configuration. $\xi(y)$ is a stationary random function of y with $\langle\xi\rangle = 0$. The whole system rotates counter-clockwise about a vertical axis directed out of the paper with angular velocity $\frac{1}{2}f$. A Poincaré wave incident at an angle θ_0 to the mean normal to the coastline generates (i) a specularly reflected Poincaré wave, (ii) diffusely scattered Poincaré waves and (iii) a trapped Kelvin wave.

in which the angular brackets denote the ensemble average. In the absence of coastal irregularities $|R| = |a|$; for $\xi \neq 0$ however, $|R| < |a|$ and the correction term is proportional to an integral over the spectrum $\Phi(m)$ of $\xi(y)$, which is defined by

$$\Phi(m) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathscr{R}(y)\, e^{-imy}\, dy, \tag{2.5}$$

where
$$\mathscr{R}(y) = \langle \xi(y+z)\,\xi(z)\rangle \tag{2.6}$$
is the covariance of ξ.

Now, for any given realization of the coast ξ, the total scattered field consists of a superposition of ϕ_R, the coherent component, and a diffusely scattered random field ϕ'. The latter is partitioned between P-wave noise and a K-wave (see figure 1). The specular reflexion coefficient R given by (2.4) is obtained by imposing the condition that the fluid velocity normal to the coast be zero:

$$u = v\xi_y \quad \text{on} \quad x = \xi(y),$$

where (u, v) are respectively the (x, y) components of velocity. Expanding the terms in this equation about $x = 0$, we find

$$u = v\xi_y - u_x\xi - \tfrac{1}{2}u_{xx}\xi^2 + v_x\xi\xi_y \quad \text{on} \quad x = 0 \tag{2.7}$$

correct to second order in ξ. Equation (2.7) is expected to provide a good approximation to the exact boundary condition provided that $K_0\langle\xi^2\rangle^{\frac{1}{2}} \ll 1$, i.e. provided that the characteristic wavelength $\sim 2\pi/K_0$ is large compared with $\langle\xi^2\rangle^{\frac{1}{2}}$. Now, as discussed at length in HM, a further simplification of (2.7) can be made by replacing terms quadratic in ξ by their ensemble averages. Thus, since $\langle\xi\xi_y\rangle = 0$, we then have

$$u = v\xi_y - u_x\xi - \tfrac{1}{2}u_{xx}\langle\xi^2\rangle \quad \text{on} \quad x = 0 \tag{2.8}$$

for sufficiently small ξ. Using the relations

$$\left.\begin{aligned} u &= g(f\phi_y - i\omega\phi_x)/(\omega^2 - f^2), \\ v &= -g(f\phi_x + i\omega\phi_y)/(\omega^2 - f^2), \end{aligned}\right\} \tag{2.9}$$

which are valid for a harmonic time dependence $e^{-i\omega t}$, (2.8) can readily be written in terms of ϕ (the total surface elevation).

3. Integral equation for the diffusely scattered field

The equation which describes conservation of total wave energy at the coast may be obtained by multiplying the boundary condition (2.8) by the perturbation pressure p evaluated on the boundary:

$$p(u - v\xi_y + u_x\xi + \tfrac{1}{2}u_{xx}\langle\xi^2\rangle) = 0 \quad \text{on} \quad x = 0, \tag{3.1}$$

where $p \equiv p(\xi, y) = p(0,y) + p_x(0,y)\,\xi + \frac{1}{2}p_{xx}(0,y)\langle\xi^2\rangle$. Since $p = \rho g\phi$ and u, u_x, u_{xx} and v can be written in terms of ϕ by means of (2.9), equation (3.1), when suitably averaged, gives the net effect of scattering from the incident field. This was essentially the procedure followed in HM to obtain expressions for the power fluxes into the K-wave and the P-wave noise. To isolate the scattering of energy into a particular wave mode, we must first decompose the scattered wave field into its constituent Fourier components ϕ_n, say. Then in the segment of the coast in which the Fourier analysis has been executed, the boundary irregularities will lead to a *redistribution* of energy amongst these components. In order to determine the effect of the irregularities on *one* of these components, (2.8) is multiplied by $p_n = \rho g\phi_n$ evaluated on the boundary $x = \xi(y)$. Thus the equation

$$\rho g\phi_n(u - v\xi_y + u_x\xi + \tfrac{1}{2}u_{xx}\langle\xi^2\rangle) = 0 \quad \text{on} \quad x = 0, \tag{3.2}$$

where $\phi_n \equiv \phi_n(\xi, y) = \phi_n(0,y) + \phi_{nx}(0,y)\,\xi + \frac{1}{2}\phi_{nxx}(0,y)\langle\xi^2\rangle$, may be taken to describe the interactions between ϕ_n and all other components of the field. Finally, an appropriate average of (3.2) will yield an integral equation for the mean-square scattered field in terms of the individual scattered modes.

To discuss this method of describing multiple scattering in more detail it is necessary to define the fundamental length scales involved. We first introduce the small parameter

$$\epsilon = \langle\xi^2\rangle^{\frac{1}{2}}/r \ll 1$$

(denoted by β in HM), which is the ratio of the amplitude of the coastal irregularities to the Rossby radius of deformation $r = s/f$. The latter is the basic horizontal length scale associated with P- and K-waves.† Then if $\lambda(=O(r))$ denotes a typical wavelength it follows that the multiple scattering of waves at the coast produces significant effects over distances of order λ/ϵ^2 (see HM; Mysak & Tang 1974). It is therefore appropriate to choose an interval of the coast of length $L \ll \lambda/\epsilon^2$ in which to Fourier analyse the wave field, and to examine the variation in the amplitudes of the Fourier coefficients over much larger distances. In a semi-infinite strip parallel to the x axis

† For the internal modes defined in § 2, r is replaced by the internal deformation radius $r_j = s_j/f$ and accordingly $\epsilon_j \gg \epsilon$. Hence for a given coastline, the internal modes are more strongly affected by the irregularities.

lying in $x > 0$ and of width L centred at $y = 0$, say, the total field may be set in the form

$$\phi = \sum_{n=-\infty}^{\infty} \{h(m_n) + R\delta_{m_n m_0}\} \exp i\{l_n x + m_n y - \omega t\} + \phi_0 + \text{c.c.}, \tag{3.3}$$

where R is defined in (2.4), $h(m_n)$ is a Fourier coefficient of the random field which satisfies $\langle h(m_n)\rangle \equiv 0$, and where $m_n = 2n\pi/L$ $(n \neq 0)$, m_0 = wavenumber component of (2.1), $l_n = (K_0^2 - m_n^2)^{\frac{1}{2}} > 0$ if $m_n^2 < K_0^2$ and is positive imaginary otherwise, δ_{ij} = Kronecker delta symbol and ϕ_0 = incident field (2.1).

In order to determine the properties of the random field in the semi-infinite strip we shall find it necessary to apply a local form of Born's scattering approximation (see below), the validity of which leads to the additional requirement that $L^2\epsilon^2/l^2 \ll 1$, where l is the correlation scale of the irregularities (Frisch 1968). Clearly this inequality holds if $L = O(l/\epsilon^{\frac{1}{2}})$. Thus, in summary, we introduce three length scales, l, L and λ/ϵ^2, and for sufficiently small ϵ we have the following ordering amongst these scales: $l \ll L \ll \lambda/\epsilon^2$.

The Fourier coefficients $h(m_n)$ in (3.3) are purely random, and it is convenient to write

$$h = \bar{h} + h', \tag{3.4}$$

where $\bar{h}$ denotes the average of h over an ensemble of realizations of ξ within the strip of width L. That is, $\bar{h}$ is that part of the random field that is locally uncorrelated with ξ: $\langle \bar{h}\xi(y)\rangle = 0$ for $y \in L$ but $\langle h'\xi(y)\rangle \neq 0$, where as before angular brackets denote the ensemble average with respect to the whole of the coast. The local Born approximation will be used to determine h', and since $l \ll L \ll \lambda/\epsilon^2$, it can be shown that $h'/\bar{h} = O(\epsilon)$ [see (3.16)].

In the interval L, we also write

$$\xi(y) = \sum_{q=-\infty}^{\infty} \tilde{\xi}(m_q) \exp(im_q y)\,\Delta m, \tag{3.5}$$

where $\Delta m = 2\pi/L$, $m_{-q} = -m_q$, and since $\xi(y)$ is real, $\tilde{\xi}(m) = \tilde{\xi}^*(-m)$. For $l \ll L$, we have

$$\overline{\tilde{\xi}(m_q)\,\tilde{\xi}(m_p)\,\Delta m} \simeq \Phi(m_q)\,\delta_{m_p,-m_q}, \tag{3.6}$$

where $\Phi(m)$ is the spectrum defined in (2.5) and the overbar denotes the average defined in the preceding paragraph. We shall be interested in the continuum limit obtained as $\epsilon \to 0$ $(L \to \infty)$, in which case the approximation (3.6) becomes exact (Stratonovich 1963) and the interval in wavenumber space between successive Fourier components tends to zero. For this reason we introduce a continuum distribution function $\beta(m)$ for the random modes, in which the subscript is discarded:

$$\langle |h(m)|^2\rangle = \beta(m)\,\Delta m, \tag{3.7}$$

where $\Delta m \to dm$ in the continuum limit. We now proceed to derive an integral equation for $\beta(m)$ which is valid asymptotically as $\epsilon \to 0$.

Before averaging (3.2) we record here for future reference the expansion

$$\phi(\xi, y) = \phi(0, y) + \phi_x(0, y)\,\xi + \tfrac{1}{2}\phi_{xx}(0, y)\,\langle\xi^2\rangle. \tag{3.8}$$

Also, we rewrite (3.2) in the form

$$\rho g\{\phi + \phi_x\xi + \tfrac{1}{2}\phi_{xx}\langle\xi^2\rangle\}_n u = \rho g\{\phi + \phi_x\xi + \tfrac{1}{2}\phi_{xx}\langle\xi^2\rangle\}_n \{v\xi_y - u_x\xi - \tfrac{1}{2}u_{xx}\langle\xi^2\rangle\} \quad \text{on} \quad x = 0. \tag{3.9}$$

From (3.3) and (3.8) it is readily shown that

$$\phi(\xi,y) = \sum_{n=-\infty}^{\infty} \{(1-\tfrac{1}{2}\langle\xi^2\rangle l_n^2)\,h(m_n) + (a+R)\,(1-\tfrac{1}{2}\langle\xi^2\rangle l_0^2)\,\delta_{m_n m_0}\} \exp[i(m_n y - \omega t)]$$
$$+ \sum_{N,q=-\infty}^{\infty} \{il_N h(m_N) - il_0(a-R)\,\delta_{m_N m_0}\}\,\tilde{\xi}(m_q)\,\Delta m \exp[i(m_q+m_N)y - i\omega t] + \text{c.c.}$$

Thus the nth Fourier component of this is

$$[\phi(\xi,y)]_n = \{[h(m_n) + (a+R)\,\delta_{m_n m_0}]\,[1-\tfrac{1}{2}\langle\xi^2\rangle l_n^2] + \sum_N [il_N h(m_N)$$
$$- il_0(a-R)\,\delta_{m_N m_0}]\,\tilde{\xi}(m_n - m_N)\,\Delta m\} \times \exp[i(m_n y - \omega t)] + \text{c.c.} \quad (3.10)$$

In (3.9) we also need expressions for u, v, u_x and u_{xx} in terms of ϕ. From (2.9) and (3.3) we find that on $x = 0$

$$\left.\begin{aligned}
u &= \frac{g}{\omega^2-f^2}\sum_n [(im_n f + \omega l_n)\,h(m_n) + A\delta_{m_n m_0}] \exp[i(m_n y - \omega t)] + \text{c.c.},\\
u_x &= \frac{ig}{\omega^2-f^2}\sum_n [l_n(im_n f + \omega l_n)\,h(m_n) - l_0 B\delta_{m_n m_0}] \exp[i(m_n y - \omega t)] + \text{c.c.},\\
u_{xx} &= \frac{-g}{\omega^2-f^2}\sum_n [l_n^2(im_n f + \omega l_n)\,h(m_n) + l_0^2 A\delta_{m_n m_0}] \exp[i(m_n y - \omega t)] + \text{c.c.},\\
v &= \frac{-g}{\omega^2-f^2}\sum_n [(il_n f - \omega m_n)\,h(m_n) - C\delta_{m_n m_0}] \exp[i(m_n y - \omega t)] + \text{c.c.},
\end{aligned}\right\} \quad (3.11)$$

where

$$\left.\begin{aligned}
A &= im_0 f(a+R) - \omega l_0(a-R),\\
B &= im_0 f(a-R) - \omega l_0(a+R),\\
C &= il_0 f(a-R) + \omega m_0(a+R).
\end{aligned}\right\} \quad (3.12)$$

Let us now define the averaging procedure to be applied to the boundary condition (3.9), in which (3.5), (3.10) and (3.11) are to be used. First integrate (3.9) with respect to y over the width L of the strip. This isolates the Fourier components $\exp\{\pm im_n y\}$ since (for $q \neq 0$) $\{\exp(im_q y)\}_{q=-\infty}^{\infty}$ is an orthogonal set on L. Then average over a wave period $2\pi/\omega$. Finally, take the average $\langle\ \rangle$ over an ensemble of realizations of the coastal irregularities and pass to the continuum limit $\epsilon \to 0$. Actually, for those interaction terms involving the product of $\tilde{\xi}$ and h it is convenient to do the ensemble averaging in two stages: first take the local average $\overline{(\cdot)}$ with respect to all possible realizations of $\xi(y)$ for $y \in L$; then average the result over all realizations of $\xi(y)$ for $y \notin L$. The details of this calculation are somewhat involved and only the main points will be discussed. The procedure is a lengthy exercise that is identical to the analogous calculation in acoustic theory described by Howe (1974*a*).

Considering first the left-hand side of (3.9), we obtain two sets of terms I_1 and I_2, say. The terms in I_1 consist of expressions proportional to $\langle|h(m_n)|^2\rangle$ and the squares of the amplitudes of the incident and specularly reflected fields; in the continuum limit I_1 is given by

$$I_1 = \frac{2\omega l\rho g^2 dm}{\omega^2-f^2}(1-\tfrac{1}{2}\langle\xi^2\rangle l^2)\{\beta(m)\,H(K_0^2-m^2) - (|a|^2-|R|^2)\,\delta(m-m_0)\}, \quad (3.13)$$

where $H(x)$ is the unit step function, $\delta(x)$ is the Dirac delta function and

$$l \equiv l(m) = (K_0^2-m^2)^{\frac{1}{2}}.$$

Also, we have used (3.7) and the relation

$$\delta_{m_n m_0} = \frac{\Delta m}{2\pi}\int_{-\frac{1}{2}L}^{\frac{1}{2}L} \exp\{i(m_n - m_0)\,x\}\,dx \to \delta(m-m_0)\,dm \quad \text{as} \quad L\to\infty.$$

The terms in I_2 involve interactions between h and $\tilde{\xi}$:

$$I_2 = \frac{\rho g^2 i\Delta m}{\omega^2 - f^2}\{(im_n f + \omega l_n)^* \sum_q l_q \overline{\langle h^*(m_n)\,\tilde{\xi}(m_n - m_q)\,h(m_q)\rangle}$$
$$- l_0(a-R)\,(im_n f + \omega l_n)^* \sum_q \delta_{m_q m_0} \overline{\langle \tilde{\xi}(m_n - m_q)\,h^*(m_q)\rangle}$$
$$+ A^* \sum_q l_q \overline{\langle \tilde{\xi}(m_n - m_q)\,h(m_q)\rangle}\,\delta_{m_n m_0}\} + \text{c.c.} \qquad (3.14)$$

In the continuum limit, $m_n \to m$ and the barred terms result in expressions of the form

$$\sum_q \Phi(m - m_q)\,\beta(m_q)\,\Delta m \to \int \Phi(m - M)\,\beta(M)\,dM.$$

To see how an expression such as $\Phi(m-m_q)\,\beta(m_q)\,\Delta m$ arises from the first barred term in (3.14) for example, we first introduce the approximation

$$\overline{\langle h^*(m_n)\,\tilde{\xi}(m_n - m_q)\,h(m_q)\rangle} \simeq \overline{\langle h'^*(m_n)\,\tilde{\xi}(m_n - m_q)\,\bar{h}(m_q)\rangle} + \overline{\langle \bar{h}^* \tilde{\xi}(m_n - m_q)\,h'(m_q)\rangle}, \qquad (3.15)$$

in which terms nonlinear in the *locally* scattered field, i.e. in $h'(m)$, are neglected, use having been made of (3.4). Then the *local Born approximation* is used to determine h' in (3.15) (see HM for details):

$$h'(m_n) = \sum_N \frac{-i\tilde{\xi}(m_n - m_N)\,\Delta m}{(im_n f + \omega l_n)}$$
$$\times \{[im_n l_N f - \omega(m_n m_N - K_0^2)]\,\bar{h}(m_N) - [(m_n - m_N)\,C + l_0 B]\,\delta_{m_N m_0}\}. \qquad (3.16)$$

In (3.16), ω is assigned a small positive imaginary part to ensure that the causality condition is satisfied by local scattering processes (Lighthill 1960). It is clear that $h'/\bar{h} = O(\epsilon)$, which was assumed *a priori* in (3.15). Thus the first term in (3.15), for example, becomes, on using (3.6) and (3.16),

$$\overline{\langle h'^*(m_n)\,\tilde{\xi}(m_n - m_q)\,\bar{h}(m_q)\rangle}$$
$$\simeq \frac{i\Phi(m_n - m_q)}{(im_n f + \omega l_n)^*}\left\{\left[\frac{-im_n f}{\omega}(\omega l_q^* - im_q f) - \left(\frac{\omega^2 - f^2}{\omega}\right)\left(m_n m_q - \frac{\omega^2}{s^2}\right)\right]\langle|\bar{h}(m_q)|^2\rangle\right.$$
$$\left. - [(m_n - m_q)\,C^* + l_0 B^*]\,\delta_{m_q m_0}\langle \bar{h}(m_q)\rangle\right\}$$
$$\simeq \frac{i\Phi(m_n - m_q)}{(im_n f + \omega l_n)^*}\left\{\frac{-im_n f}{\omega}(\omega l_q^* - im_q f) - \frac{(\omega^2 - f^2)}{\omega}\left(m_n m_q - \frac{\omega^2}{s^2}\right)\right\}\langle|h(m_q)|^2\rangle$$
$$\to \frac{i\Phi(m - m_q)}{(imf + \omega l)^*}\left\{\frac{-imf}{\omega}(\omega l_q^* - im_q f) - \left(\frac{\omega^2 - f^2}{\omega}\right)\left(mm_q - \frac{\omega^2}{s^2}\right)\right\}\beta(m_q)\,\Delta m.$$

To arrive at the penultimate line above, we have used the fact that $\bar{h} = h(1 + O(\epsilon))$ and the properties of the ensemble average stated after (3.4). Calculations along similar lines are performed for the remaining barred terms in (3.14) and for the interaction terms which arise when averaging the right-hand side of (3.9).

In executing the above programme we discover that one set of terms is associated with the coherent field and corresponds to the 'determinate' part of the product on the left-hand side of (3.2), viz.

$$\langle p_n\rangle\{\langle u\rangle-\langle v\xi_y\rangle\}=0. \tag{3.2a}$$

All such 'interactions' are associated with the mean field and appear multiplied by $\delta(m-m_0)$ in the continuum limit. Since (3.2*a*) holds independently of the random interactions (see HM) we can delete from the average of (3.9) all terms proportional to $\delta_{m_n m_0}$ or $\delta(m-m_0)$. Thus, for example, we delete the second term within the curly brackets in (3.13) and the third summation in (3.14). The remaining sets of terms describe the characteristics of the scattered, incoherent wave field. In the continuum limit we obtain in this way the following *integral equation* for the distribution function $\beta(m)$:

$$\begin{aligned}\frac{2\rho g^2\omega l(m)}{\omega^2-f^2}\,\beta(m)\,H(K_0^2-m^2)\\ =\frac{\rho g^2}{\omega^2-f^2}\int_{-\infty}^{\infty}\Phi(m-M)\,[\beta(M)\,\mu(m)\,\Gamma(m,M)-\beta(m)\,\mu(M)\,\Gamma(M,m)]\,dM\\ +\frac{\rho g^2}{\omega^2-f^2}\,|(m-m_0)\,C+l_0\,B|^2\,\Phi(m-m_0)\,\mu(m),\end{aligned} \tag{3.17}$$

where

$$\begin{aligned}\mu(m)&=\frac{1}{imf+\omega l(m)}+\text{c.c.}\\ &=\frac{2\pi f}{\omega^2-f^2}\,\delta\left(m+\frac{\omega}{s}\right)+\frac{2\omega l(m)}{\omega^2 l^2(m)+m^2f^2}\,H(K_0^2-m^2),\end{aligned} \tag{3.18}$$

$$\Gamma(m,M)=\left|\frac{imf}{\omega}(iMf+\omega l(M))-\frac{(\omega^2-f^2)}{\omega}\left(mM-\frac{\omega^2}{s^2}\right)\right|^2 \tag{3.19}$$

and the second line of (3.18) follows from the preceding one as $\operatorname{Im}\omega\to+0$. When $f=0$ (no rotation, in which case the equations for sound waves and long waves are identical), it is easy to check that (3.17) reduces to the integral equation derived by Howe (1974*a*) [cf. his equation (3.25)].

4. Physical interpretation of the integral equation

Let us write (3.17) in the form

$$\begin{aligned}&\frac{\rho g^2}{\omega^2-f^2}\left\{2\omega lH(K_0^2-m^2)+\int\Phi(m-M)\,\mu(M)\,\Gamma(M,m)\,dM\right\}\beta(m)\\ &\quad=\frac{\rho g^2}{\omega^2-f^2}\mu(m)\left\{\int\Phi(m-M)\,\beta(M)\,\Gamma(m,M)\,dM+|(m-m_0)\,C+l_0\beta|^2\,\Phi(m-m_0)\right\}.\end{aligned} \tag{4.1}$$

The second term in the brace brackets on the right of this equation represents the flux of energy per unit area of coast from the incident coherent wave m_0 into mode m

of the random field. Since this term is proportional to $\mu(m)$, it is apparent from (3.18) that on average energy enters a single K-wave at $m = -\omega/s$ and a continuum of P-waves satisfying $m^2 < K_0^2$. Further, the right-hand side of (4.1) is proportional to $\mu(m)$ and the left-hand side to $\beta(m)$, from which we deduce that $\beta(m) \neq 0$ only for these modes.

Now the first term on the left of (4.1), being proportional to $\langle p_m u_m \rangle$, represents the rate at which the P-mode m is radiating energy into the open ocean ($x > 0$) per unit area of the coast [cf. Howe 1974*a*, equation (3.29)]. The second term on the left is proportional to the rate at which the (P or K) mode m is losing energy by scattering into all other (P or K) wave modes M. These decay mechanisms are balanced on the right-hand side by the flux of energy into the mode m due to (i) the multiple scattering of all other random modes M and (ii) the direct scattering of energy from the coherent field.

When (4.1) is integrated over all values of m it follows from the above that the component of the power flux into the open ocean which is associated with the random wave modes is given by

$$\begin{aligned} P_S &\equiv \frac{2\rho g^2 \omega}{\omega^2 - f^2} \int l(m)\,\beta(m)\,H(K_0^2 - m^2)\,dm \\ &= \frac{\rho g^2}{\omega^2 - f^2} \iint \Phi(m-M)\,[\beta(M)\,\mu(m)\,\Gamma(m,M) - \beta(m)\,\mu(M)\,\Gamma(M,m)]\,dM\,dm \\ &\quad + \frac{\rho g^2}{\omega^2 - f^2} \int |(m-m_0)\,C + l_0 B|^2\,\Phi(m-m_0)\,\mu(m)\,dm. \end{aligned}$$

The double integral on the right vanishes because of the asymmetry of the integrand, and the integral involving $\Phi(m-m_0)$ may be simplified by use of (2.4), (3.12) and (3.18). Thus correct to the second order in ξ we can set

$$P_S = 4|a|^2 \{\langle P_R \rangle_K + \langle P_R \rangle_P\}, \tag{4.2}$$

where
$$\langle P_R \rangle_K = \frac{2\pi f \rho g \omega^2 l_0^2}{d(\omega^2 - f^2)} \frac{(\omega/s + m_0)^2}{(\omega^2/s^2 - m_0^2)}\,\Phi\left(\frac{\omega}{s} + m_0\right)$$

and
$$\langle P_R \rangle_P = \frac{2\rho g^2 l_0^2 \omega}{(\omega^2/s^2 - m_0^2)(\omega^2 - f^2)} \int_{-K_0}^{K_0} \frac{\Phi(m-m_0)\,(m m_0 - \omega^2/s^2)^2\,|K_0^2 - m^2|^{\frac{1}{2}}}{\omega^2/s^2 - m^2}\,dm.$$

Here $\langle P_R \rangle_K$ and $\langle P_R \rangle_P$ are respectively the expressions obtained in HM [equations (4.11) and (4.12)] for the total power flux scattered from the incident wave into the K-wave and the P-wave noise. The factor $4|a|^2$ appears in (4.2) because the incident wave has amplitude $2a$ rather than unity, as was the case in HM. We conclude that (3.17) describes the detailed interactions at the coast between individual wave modes in a manner that satisfies conservation of total wave energy.

To see how multiple scattering affects the partition of energy between the K- and P-modes, let us write

$$\beta(m) = I\delta(m + \omega/s) + S(m)\,H(K_0^2 - m^2). \tag{4.3}$$

Substituting (4.3) into (3.17) and integrating over a small interval about $m = -\omega/s$ gives

$$I\frac{2\omega\rho g^2(\omega^2-f^2)}{s^2}\int_{-K_0}^{K_0}\frac{l(M)\,\Phi(M+\omega/s)\,(M+\omega/s)^2}{\omega^2l^2(M)+M^2f^2}\,dM$$

$$=\frac{2\pi f\rho g^2}{(\omega^2-f^2)^2}\int_{-K_0}^{K_0}\Phi(M+\omega/s)\,S(M)\left|-\frac{if}{s}(iMf+\omega l(M))+\frac{(\omega^2-f^2)}{s}\left(M+\frac{\omega}{s}\right)\right|^2dM$$

$$+|(m_0+\omega/s)\,C-l_0B|^2\frac{2\pi f\rho g^2}{(\omega^2-f^2)^2}\Phi(m_0+\omega/s). \quad (4.4)$$

This equation establishes the equality between the power flux from the K-wave into the random P-modes (left-hand side) and the flux of energy into the K-wave from multiply scattered P-modes M and the directly scattered coherent field. Observe that (4.4) contains both of the unknown amplitude functions I and $S(m)$; a second equation is obtained by substituting (4.3) into (3.17) and retaining only those terms which are non-zero for $m^2 < K_0^2$:

$$\frac{2\rho g^2\omega l(m)}{\omega^2-f^2}S(m)=\frac{2\omega l\rho g^2(\omega^2-f^2)\,(m+\omega/s)^2}{s^2(\omega^2l^2+m^2f^2)}I\Phi(m+\omega/s)$$

$$+\frac{2\omega l\rho g^2}{(\omega^2-f^2)\,(\omega^2l^2+m^2f^2)}\int_{-K_0}^{K_0}S(M)\,\Phi(m-M)\,\Gamma(m,M)\,dM$$

$$+|(m-m_0)\,C+l_0B|^2\frac{2\rho g^2\omega l(m)}{(\omega^2-f^2)\,(\omega^2l^2+m^2f^2)}\Phi(m-m_0)$$

$$-\frac{2\pi f\rho g^2}{(\omega^2-f^2)^2}\left|-\frac{if}{s}(imf+\omega l)+K_0^2s\left(m+\frac{\omega}{s}\right)\right|^2S(m)\,\Phi(m+\omega/s)$$

$$-\frac{2\omega lg^2}{\omega^2-f^2}\int_{-K_0}^{K_0}\frac{l(m)\,\Gamma(M,m)\,\Phi(m-M)}{\omega^2l^2(M)+M^2f^2}\,dMS(m). \quad (4.5)$$

This equates the flux of energy into the open ocean ($x > 0$) in the random P-wave mode m to the difference between (i) the power scattered into that mode from the K-wave, other P-wave modes M and the coherent field and (ii) the power lost through re-scattering into the K- and P-waves. Since the spectrum function $\Phi\propto\langle\xi^2\rangle$ [see (2.5) and (2.6) above], (4.5) predicts that $S(m)$ is also of order $\langle\xi^2\rangle$. The first term on the right of (4.4) is therefore $O(\xi^4)$, and is small compared with the second, $O(\xi^2)$ component describing the interaction of the coherent field with the coast. Thus in a first approximation, (4.4) implies that

$$I\simeq\frac{\pi fs^2}{\omega(\omega^2-f^2)^3}\frac{|(m_0+\omega/s)\,C-l_0A|^2\,\Phi(m_0+\omega/s)}{\displaystyle\int_{-K_0}^{K_0}\frac{l(M)\,\Phi(M+\omega/s)\,(M+\omega/s)^2\,dM}{\omega^2l^2(M)+M^2f^2}}$$

$$\simeq\frac{4\pi fl_0^2\,\omega s^{-2}(\omega/s+m_0)(\omega/s-m_0)^{-1}\Phi(m_0+\omega/s)}{K_0^2\displaystyle\int_{-K_0}^{K_0}(\omega/s+M)\,(\omega/s-M)^{-1}\Phi(M+\omega/s)|K_0^2-M^2|^{\frac{1}{2}}dM}, \quad (4.6)$$

where in the definitions (3.12) of C and B use has been made of (2.4) with the $O(\xi^2)$ correction term neglected. The appearance of the spectrum Φ in both the denominator and the numerator of (4.6) reveals that the strength of the Kelvin wave is of the same order as that of the incident P-wave and independent of the magnitude of the coastal irregularities. In particular this indicates that, under steady-state conditions, the Kelvin-wave amplitude greatly exceeds that of the P-wave noise, for which $S(m) = O(\xi^2)$.

This is a plausible conclusion which arises because the K-wave is confined to the immediate vicinity of the coast, where the rates at which it gains and loses energy are both $O(\xi^2)$. The assumption of steady-state, harmonic conditions implies that the K-wave has had an infinite amount of time in which to acquire its finite amplitude. However, a quasi-steady incident field of infinite extent will not be encountered in practice. Accordingly the problem of determining the manner in which these simple results are modified when account is taken of the space–time variations of the incident field must now be considered.

5. The kinetic equation

Equation (3.17) for the continuum distribution function $\beta(m)$ has been derived on the assumption that the statistical properties of the local random Fourier coefficients $h(m)$ of (3.3) depend neither on the time nor on the position along the coast, i.e. (3.17) describes multiple scattering of an incident plane wave of infinite extent under 'equilibrium' conditions. Space–time variations in $h(m)$ (and hence $\beta(m)$) caused by scattering will scale respectively on distances/times $O(1/\epsilon^2)$ greater than the characteristic wavelength/wave period. In other words the derivatives of $h(m)$ with respect to y and t are $O(\epsilon^2)$ quantities and therefore small for small coastal irregularities ξ. On this basis we shall now derive additional terms for inclusion in (3.17) which take account of possible space–time variations in $\beta(m)$ due to scattering. This will lead to an integro-differential kinetic equation (cf. Howe 1974*b*; Mysak & Howe 1976).

In § 4 it was deduced that, in the extreme case of an incident plane wave of infinite extent, for which the energy scattered into the random modes must be maximal, the K- and P-wave energy densities are respectively $O(1)$ and $O(\xi^2)$ quantities. If the amplitudes of these waves are slowly modulated in the manner described above, it follows that the space/time derivatives of the K- and P-components of $\beta \equiv \beta(m, y, t)$ are respectively $O(\xi^2)$ and $O(\xi^4)$. Since the leading terms in (3.17) are already $O(\xi^2)$ it is clear that only the derivatives of the K-wave need be taken into account. Further it is necessary to consider only the additional terms which arise in (3.17) from averaging $p_n u \equiv \rho g \phi_n u$ on the left of (3.9), since the remaining terms in this equation involve ξ and ξ^2 and would give contributions of higher order than ξ^2.

Consider a single component of the *unmodulated* scattered field:

$$\phi = h(m) \exp i\{lx + my - \omega t\} + \text{c.c.}, \tag{5.1}$$

where $l = l(m) = [K_0^2 - m^2]^{\frac{1}{2}} > 0$ when $m^2 < K_0^2$ and is positive imaginary otherwise. Using (5.1) in (2.9) we have

$$u = \frac{g}{\omega^2 - f^2} \{ifm + \omega l(m)\} \phi = \mathscr{L}(m, \omega) \phi \equiv \mathscr{L}(-i\partial_y, i\partial_t) \phi, \text{ say.} \tag{5.2}$$

Suppose now that h is a slowly varying function of position y and time t:

$$h = h(m, y, t), \tag{5.3}$$

where $h_y, h_t = O(\epsilon^2)\,h$. Then in place of (5.2) we shall have

$$\begin{aligned} u &= \exp i(lx + my - \omega t)\mathscr{L}(m - i\partial_y, \omega + i\partial_t)\,h + \text{c.c.} \\ &\simeq \exp i(lx + my - \omega t)\{\mathscr{L}(m, \omega) + i[\mathscr{L}_\omega \partial_t - \mathscr{L}_m \partial_y]\}\,h + \text{c.c.} \end{aligned} \tag{5.4}$$

This result indicates that in evaluating the mean value $\langle \rho g \phi_n u\rangle$ of the left-hand side of (3.9) the additional terms will arise from the second component of the curly brackets in (5.4). Recalling that it is necessary to account only for the derivatives of the K-wave amplitude for which $m = -\omega/s$, it follows easily that

$$i[\mathscr{L}_\omega \partial_t - \mathscr{L}_m \partial_y]_{m=-\omega/s} = (g/sf)(\partial_t - s\partial_y). \tag{5.5}$$

Since $s = (gd)^{\frac{1}{2}}$ is the group velocity of a K-wave we see that (5.5) represents the usual space-time evolution operator for a slowly modulated wave packet. Hence the term I_1 corresponding to $\langle \rho g \phi_n u\rangle$ involves the following contribution in addition to those already shown in (3.13):

$$\delta_{m,-\omega/s}\frac{\rho g^2}{sf}(\partial_t - s\partial_y)\,\beta(m, y, t)\,dm. \tag{5.6}$$

Incorporating this into (3.17) we obtain the desired integro-differential (kinetic) equation for $\beta(m, y, t) \equiv \beta(m)$:

$$\begin{aligned} &\frac{\rho g^2}{sf}(\partial_t - s\partial_y)\,\beta(m)\,\delta_{m,-\omega/s} + \frac{2\rho g^2\omega l}{\omega^2 - f^2}H(K_0^2 - m^2)\,\beta(m) \\ &\quad = \frac{\rho g^2}{\omega^2 - f^2}\int_{-\infty}^{\infty}\Phi(m - M)\{\beta(M)\,\mu(m)\,\Gamma(m, M) - \beta(m)\,\mu(M)\,\Gamma(M, m)\}\,dM \\ &\qquad + \frac{\rho g^2}{\omega^2 - f^2}|(m - m_0)\,C + l_0 B|^2\,\Phi(m - m_0)\,\mu(m). \end{aligned} \tag{5.7}$$

Note that the additional 'propagation' term which now appears in this equation will make a non-trivial modification to the multiple scattering analysis only if a space/time variation is introduced into the original problem of the scattering of the incident Poincaré wave (2.1). Accordingly we shall assume that the amplitude coefficient a of that wave is a function of y, t which varies slowly on a scale of wavelength/wave period.

Multiply (5.7) by the mean depth d and use the decomposition (4.3) of $\beta(m)$ to obtain

$$\begin{aligned} &\left[\frac{2\rho g^2 d\omega l}{\omega^2 - f^2}S(m)\right]H(K_0^2 - m^2) + (\partial_t - s\partial_y)\left[\frac{\rho g^2 d}{gf}I(y, t)\right]\delta\left(m + \frac{\omega}{s}\right) \\ &\quad = \frac{\rho g^2 d}{\omega^2 - f^2}\int\Phi(m - M)\,[\beta(M)\,\mu(m)\,\Gamma(m, M) - \beta(m)\,\mu(M)\,\Gamma(M, m)]\,dM \\ &\qquad + \frac{\rho g^2 d}{\omega^2 - f^2}|(m - m_0)\,C + l_0 B|^2\,\Phi(m - m_0)\,\mu(m). \end{aligned} \tag{5.8}$$

The Fourier amplitude $h(m)$ has the dimensions of length, therefore $\beta(m)$ as defined in (3.7) has the dimensions (length)^2 per unit wavelength. It follows from (4.3) that I and $S(m)$ have dimensions (length)^2 and (length)^2 per unit wavelength respectively.

Introduce the following definitions of the terms in the square brackets on the left of (5.8):

$$P(m) \equiv \frac{2\rho g^2 d\omega l(m)}{\omega^2-f^2}\,S(m), \quad K(y,t) \equiv \frac{\rho g^2 d}{sf}\,I(y,t). \qquad (5.9),\ (5.10)$$

Then

$$\int_{-K_0}^{K_0} P(m)\,dm$$

represents the total mean power per unit length of the coast scattered into the random P-waves, and $K(y,t)$, or equivalently,

$$\int_{-\infty}^{\infty} K(y,t)\,\delta(m+\omega/s)\,dm,$$

is the mean energy per unit length of coast associated with the Kelvin wave. For example, the latter identification may be established by observing that (5.1), with $m = -\omega/s$, and (2.9) imply that

$$\phi = h(-\omega/s)\exp\{-fx/s - i\omega(y/s+t)\} + \text{c.c.},$$

$$u \equiv 0, \quad v = (-g/s)\,h(-\omega/s)\exp\{-fx/s - i\omega(y/s+t)\} + \text{c.c.}$$

Hence, to leading order, the kinetic energy per unit length of coast is given by

$$\left\langle \tfrac{1}{2}\rho\int_0^d dz\int_0^\infty v^2\,dx\right\rangle_{\text{cycle}} = \tfrac{1}{2}\rho d\,\frac{g^2}{s^2}\,\frac{4|h|^2}{2}\int_0^\infty e^{-2fx/s}\,dx$$

$$= \frac{\rho g^2 d}{2sf}\,|h|^2 = \frac{\rho g^2 d}{2sf}\,I. \qquad (5.11)$$

Further, the potential energy per unit length of coast is

$$\left\langle \tfrac{1}{2}\rho g\int_0^\infty \phi^2\,dx\right\rangle_{\text{cycle}} = \frac{\rho g s}{2f}\,|h|^2 = \frac{\rho g^2 d}{2fs}\,I. \qquad (5.12)$$

Combining (5.11) and (5.12) leads precisely to the definition (5.10) of $K(y,t)$.

It is convenient to record here the explicit form of (5.8) when use is made of the definitions (5.9) and (5.10) in the left-hand side:

$$P(m)\,H(K_0^2-m^2) + (\partial_t - s\partial_y)\,K\delta\left(m+\frac{\omega}{s}\right)$$

$$= \frac{g^2 d}{\omega^2-f^2}\int \Phi(m-M)\,[\beta(M)\,\mu(m)\,\Gamma(m,M) - \beta(m)\,\mu(M)\,\Gamma(M,m)]\,dM$$

$$+\frac{\rho g^2 d}{\omega^2-f^2}\,|(m-m_0)\,C + l_0 B|^2\,\Phi(m-m_0)\,\mu(m). \qquad (5.13)$$

In this result $\beta(m)$ is given by

$$\beta(m) = \frac{f}{\rho g s}\,K\delta\left(m+\frac{\omega}{s}\right) + \frac{\omega^2-f^2}{2\rho\omega l(m)\,g s^2}\,P(m)\,H\left\{\frac{\omega^2-f^2}{s^2} - m^2\right\}. \qquad (5.14)$$

Integration of (5.13) over all values of m yields an energy conservation principle analogous to (4.2).

6. Equations for $K(y,t)$ and $P(m,y,t)$

The general kinetic equation (5.13) may be split into separate but coupled equations for the distribution functions $K(y,t)$ and $P(m,y,t)$ in the manner already discussed in §4. Thus the K-equation is obtained by substituting from (5.14) into (5.13) and integrating over a small interval about $m = -\omega/s$:

$$\left(\frac{\partial}{\partial t} - s\frac{\partial}{\partial y}\right)K + \frac{1}{\tau}K = J + S_K, \tag{6.1}$$

where
$$\frac{1}{\tau} = \frac{2\omega f(\omega^2-f^2)}{s}\int_{-K_0}^{K_0}\frac{l(M)\,\Phi(M+\omega/s)\,(M+\omega/s)^2}{\omega^2 l(M)^2 + M^2 f^2}\,dM, \tag{6.2}$$

$$J = \frac{\pi f}{\omega(\omega^2-f^2)}\int_{-K_0}^{K_0}\frac{1}{l(M)}\,\Phi\left(M+\frac{\omega}{s}\right)P(M)\left|\frac{-if}{s}(iMf+\omega l(M)) + \frac{\omega^2-f^2}{s}\left(M+\frac{\omega}{s}\right)\right|^2 dM, \tag{6.3}$$

$$S_K = \frac{2\pi f\rho g s^2}{(\omega^2-f^2)^2}\left|(\omega/s+m_0)\,C - l_0 B\right|^2\Phi(m_0+\omega/s). \tag{6.4}$$

The left-hand side of (6.1) describes the decay of the Kelvin wave as it propagates without dispersion along the coast at velocity $-s$. The decay arises from the scattering of energy into a diffuse field of P-waves and occurs over a characteristic time τ. It is countered by the terms J and S_K on the right-hand side, which respectively represent the energy flux into the K-wave due to multiple scattering of the P-waves and the direct scattering of the incident Poincaré wave.

Use of the relation $l^2(M) = K_0^2 - M^2$ reduces (6.2) and (6.3) to

$$\frac{1}{\tau} = 2f\frac{\omega}{s}\int_{-K_0}^{K_0}\Phi\left(M+\frac{\omega}{s}\right)\frac{\omega/s+M}{\omega/s-M}\,l(M)\,dM, \tag{6.2a}$$

$$J = \frac{\pi f\omega}{s^2}\int_{-K_0}^{K_0}\frac{1}{l(M)}\,\Phi\left(M+\frac{\omega}{s}\right)\left(\frac{\omega}{s}+M\right)^2 P(M)\,dM. \tag{6.3a}$$

Also (3.12) and the non-random expression (2.4) for R imply that (6.4) can be written in the form

$$S_K = \frac{4\pi f\omega l_0 P^*}{s^2}\,\Phi\left(m_0+\frac{\omega}{s}\right)\frac{\omega/s+m_0}{\omega/s-m_0}. \tag{6.4a}$$

In this expression P^* is the power in the incident wave per unit length of coast, viz. [cf. (5.9)]

$$P^* \equiv d\langle\rho g\phi_0 u_0\rangle_{\text{cycle}} = \frac{2\rho g s^2\omega l_0}{\omega^2-f^2}|a|^2, \tag{6.5}$$

in which $|a|^2 \equiv |a|^2(y,t)$ varies slowly over distances/times large compared with the characteristic wavelength/wave period.

Let us now introduce a polar representation of the incident and scattered wave fields and obtain an alternative set of expressions for $1/\tau$, J and S_K. For the incident wave write

$$(l_0, m_0) = K_0(\cos\theta_0, \sin\theta_0), \quad |\theta_0| < \tfrac{1}{2}\pi, \tag{6.6}$$

where K_0 is given by (2.2) and θ_0 is measured clockwise from the x axis (see figure 2). Frequency is conserved on scattering, so that the real wavenumber vector of a scattered P-wave may be similarly expressed:

$$\left.\begin{aligned}(l(m), m) &= K_0(\cos\theta, \sin\theta), \quad |\theta| < \tfrac{1}{2}\pi,\\ (l(M), M) &= K_0(\cos\phi, \sin\phi), \quad |\phi| < \tfrac{1}{2}\pi,\end{aligned}\right\} \tag{6.7}$$

where θ and ϕ are measured counter-clockwise from the x axis. Substitution of (6.6) and (6.7) into (6.2a)–(6.4a) then leads to

$$\frac{1}{\tau} = 2fb^2\left(\frac{\omega}{s}\right)^3\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\Phi\left[\frac{\omega}{s}(1+b\sin\phi)\right]\frac{1+b\sin\phi}{1-b\sin\phi}\cos^2\phi\,d\phi, \tag{6.2b}$$

$$J = \frac{\pi f}{\omega b}\left(\frac{\omega}{s}\right)^3\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\Phi\left[\frac{\omega}{s}(1+b\sin\phi)\right](1+\alpha\sin\phi)^2\frac{Q(\phi)}{\cos\phi}\,d\phi, \tag{6.3b}$$

$$S_K = \frac{4\pi fb}{s}\left(\frac{\omega}{s}\right)^2\left(\frac{1+b\sin\theta_0}{1-b\sin\theta_0}\right)\cos\theta_0 P^*\Phi\left[\frac{\omega}{s}(1+b\sin\theta_0)\right], \tag{6.4b}$$

where

$$b = (1-f^2/\omega^2)^{\frac{1}{2}}, \tag{6.8}$$

$$Q(\phi) = P(M)\cos\phi[(\omega^2-f^2)/s^2]^{\frac{1}{2}}. \tag{6.9}$$

Note that Q and P^* have the same dimensions and that $Q(\phi)\,d\phi$ is the P-wave power per unit length of coast which radiates in the angular element $(\phi, d\phi)$ into the open ocean.

The equation for $P(m) \equiv P(m,y,t)$, and hence also that for $Q(\phi)$, is obtained by inserting (5.14) into the general kinetic equation (5.13) and specifying that $m^2 < K_0^2$; this gives

$$\begin{aligned}P(m) = {}& \frac{2\omega l(m) fsK}{(\omega^2-f^2)^2(\omega^2/s^2-m^2)}\Gamma(m, -\omega/s)\,\Phi(m+\omega/s)\\ &+\frac{l(m)}{(\omega^2-f^2)(\omega^2/s^2-m^2)}\int_{-K_0}^{K_0}\frac{1}{l(M)}\Phi(m-M)\,P(M)\,\Gamma(m,M)\,dM\\ &-\frac{P(m)}{2\omega l(m)}\int_{-K_0}^{K_0}\Phi(m-M)\,\mu(M)\,\Gamma(M,m)\,dM+S_p \quad (m^2 < K_0^2),\end{aligned} \tag{6.10}$$

where

$$S_p = \frac{2\omega l(m)\rho g^2 d}{(\omega^2-f^2)(\omega^2 l^2(m)+m^2f^2)}|(m-m_0)C+l_0B|^2\,\Phi(m-m_0). \tag{6.11}$$

As in the case of the K-equation, (6.10) and (6.11) may be simplified and then expressed in terms of angular co-ordinates. We shall not describe the straightforward details of this reduction, but present the final result:

$$\begin{aligned}Q(\theta) = {}& K2fb^2\left(\frac{\omega}{s}\right)^3\frac{1+b\sin\theta}{1-b\sin\theta}\Phi\left[\frac{\omega}{s}(1+b\sin\theta)\right]\cos^2\theta\\ &-Q(\theta)\frac{\pi f}{b\omega\cos\theta}\left(\frac{\omega}{s}\right)^3(1+b\sin\theta)^2\,\Phi\left[\frac{\omega}{s}(1+b\sin\theta)\right]\\ &+b^3\left(\frac{\omega}{s}\right)^3\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\Phi\left[\frac{\omega}{s}b(\sin\theta-\sin\phi)\right]\{F(\theta,\phi)\,Q(\phi)-F(\phi,\theta)\,Q(\theta)\}\,d\phi\\ &+4b^3\left(\frac{\omega}{s}\right)^3P^*\frac{\cos\theta_0\cos^2\theta\,\omega^4(1-b^2\sin\theta\sin\theta_0)^2}{(\omega^2\cos^2\theta_0+f^2\sin^2\theta_0)(\omega^2\cos^2\theta+f^2\sin^2\theta)}\Phi\left[\frac{\omega}{s}b(\sin\theta-\sin\theta_0)\right],\end{aligned} \tag{6.12}$$

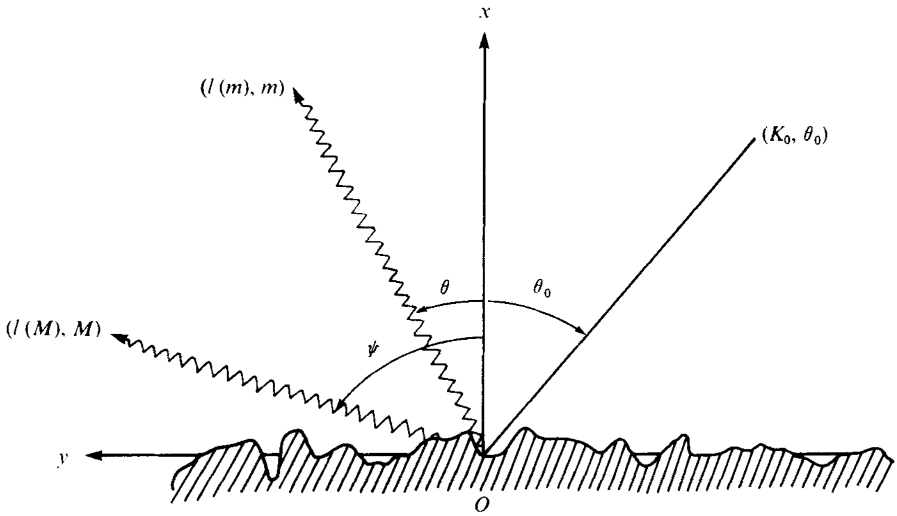

FIGURE 2. Co-ordinates for the incident and scattered fields.

where $$F(\theta,\phi) = \frac{[\omega^2(1-\sin\theta\sin\phi)^2+f^2\sin^2\theta\cos^2\phi]\cos^2\theta}{(\omega^2\cos^2\theta+f^2\sin^2\theta)\cos\phi}. \tag{6.13}$$

The terms on the right of (6.12) have the following interpretation. The power per unit length of coast scattered into that P-wave which radiates at the angle θ into the open ocean results from the following interaction processes:

(i) scattering into the P-wave from the K-wave;

(ii) re-scattering from the P-wave into the K-wave;

(iii) net scattering into the P-wave from all other random P-waves radiating at angles ϕ;

(iv) scattering into the P-wave from the incident Poincaré wave.

7. The decay of an unforced Kelvin wave

Our first application of the kinetic theory is to a situation in which the incident Poincaré wave is absent ($P^* \equiv 0$) and in which it is assumed that at time $t = 0$ there exists a Kelvin wave of frequency ω located near the origin $y = 0$. This initial condition can be modelled by
$$K(y,0) = I_0(\omega)\,\delta(y) \tag{7.1}$$
and may be regarded as characterizing a tidal crest that arrives at $t = 0$ at the mouth of a narrow canal flowing into the ocean at $y = 0$. In order to examine the subsequent propagation of the K-wave along the irregular coastline consider (6.1), in which we set $S_K \equiv 0$ and the $O(\xi^4)$ contribution J from multiply scattered P-waves is neglected:
$$\left(\frac{\partial}{\partial t} - s\frac{\partial}{\partial y}\right)K + \frac{K}{\tau} = 0. \tag{7.2}$$
The solution $$K = I_0\,\delta(y+st)\,e^{-t/\tau} \tag{7.3}$$
describes a K-wave packet propagating at speed s in the $-y$ direction and decaying in a characteristic time τ. If $0 < \omega/f < 1$, so that K_0 is pure imaginary, $\tau = \infty$ and the

wave does not decay through radiation damping because P-waves cannot propagate at these frequencies. For frequencies exceeding f, $\tau \equiv \tau(\omega, \xi^2) = O(1/\langle\xi^2\rangle)$ and is given by (6.1).

The manner in which τ varies with the radian frequency ω of the K-wave depends on the statistical properties of the coastal irregularities. For the sake of definiteness, consider here and throughout the remainder of this paper a Gaussian coastline for which the covariance takes the form

$$\mathscr{R}(y) = \langle\xi^2\rangle \exp(-y^2/l^2), \tag{7.4}$$

where l is the correlation length of $\xi(y)$. Inserting (7.4) into the definition (2.5) of the spectrum, we have

$$\Phi(m) = \Phi_0 \exp(-\tfrac{1}{4}m^2l^2) \quad (\Phi_0 = l\langle\xi^2\rangle/2\pi^{\frac{1}{2}}). \tag{7.5}$$

(In HM the notation $\mu = 2/l$ was employed.)

Before using (7.5) to compute the detailed behaviour of τ it is instructive to examine its variation in the asymptotic limit $c^2 \equiv (\omega l/2s)^2 \gg 1$, i.e. when the scale l of the coastal irregularities greatly exceeds the K-wavelength. In this case it follows from (6.2b) and (7.5) that the main contribution to the integral defining $1/\tau$ is from the small interval of ϕ in which

$$\Phi\{(\omega/s)(1+b\sin\phi)\} = \Phi_0 \exp\{-c^2(1+b\sin\phi)^2\}$$

attains its largest values. Since $0 < b < 1$, this occurs near $\phi = -\frac{1}{2}\pi$. Hence setting $\phi = -\frac{1}{2}\pi + \lambda$, we find that for $c^2 \gg 1$

$$\Phi[(\omega/s)(1+b\sin\theta)] \simeq \Phi_0 \exp[-c^2(1-b)^2 - b(1-b)c^2\lambda^2].$$

Hence we have approximately from (6.2b)

$$\frac{1}{\tau} \simeq \frac{f\omega^3 b^2(1-b)\,\Phi_0 \exp\{-c^2(1-b)^2\}}{s^3(1+b)} \int_0^\infty \lambda^2 \exp\{-b(1-b)c^2\lambda^2\}\,d\lambda \tag{7.6}$$

provided that b is not close to zero ($\omega = f$) or unity. Performing the integration we find

$$\frac{1}{\tau} \simeq \frac{f}{(1+b)}\left(\frac{b}{1-b}\right)^{\frac{1}{2}} \frac{\langle\xi^2\rangle}{l^2} \exp\{-c^2(1-b)^2\}. \tag{7.7}$$

Thus for a slowly varying coastline ($\omega l/s \gg 1$) and intermediate frequencies, $1/\tau$ is exponentially small compared with the Coriolis parameter f, i.e. τ is effectively infinite. For higher frequencies the argument of the exponential in (7.7) can be $O(1)$ even when the coefficient $b(1-b)c^2$ in the exponential of the integrand of (7.6) is large. In this case $\tau = O(l^2/f\langle\xi^2\rangle)$, which is large, but finite, and in particular indicates that at high frequencies a slowly varying coastline is characterized by a relaxation time which is comparable with that associated with tidal linear bottom friction, $\tau_B = O(10^2/f)$ (cf. Heaps 1969).

Figure 3 illustrates the variation of τ over a wide range of frequencies and for several values of $\alpha = l/2r$ (r being the Rossby radius s/f introduced in § 3). A quantity

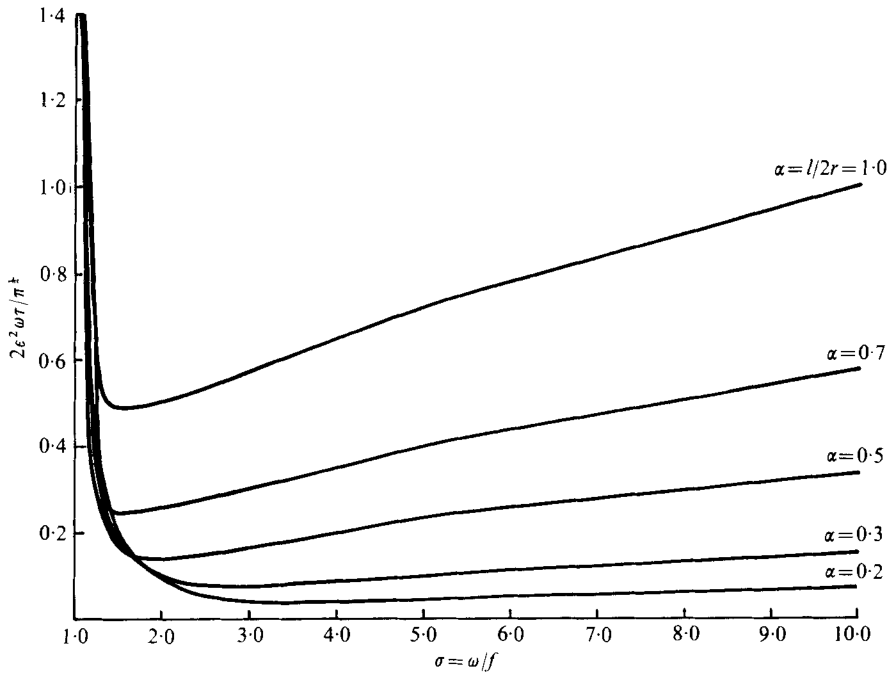

FIGURE 3. The non-dimensional e-folding distance $2\epsilon^2\omega\tau/\pi^{\frac{1}{2}}$ given by (7.8*a*) for a K-wave propagating along a coastline whose irregularities are characterized by the Gaussian spectrum (7.5).

is plotted which is numerically proportional to the e-folding or dissipation distance $s\tau$ non-dimensionalized by the wavelength, viz.

$$(4\pi^{\frac{1}{2}}\epsilon^2)\, s\tau/(s2\pi/\omega) \equiv 2\epsilon^2\omega\tau/\pi^{\frac{1}{2}}$$

$$= 1\Big/ (\sigma^2-1)\,\alpha \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \exp\left[-\sigma^2\alpha^2(1+b\sin\theta)^2\right] \frac{1+b\sin\theta}{1-b\sin\theta}\cos^2\theta\, d\theta, \tag{7.8a}$$

where
$$\epsilon = \langle\xi^2\rangle^{\frac{1}{2}}/r, \quad \sigma = \omega/f. \tag{7.8b}$$

In (7.8*b*), ϵ is the coastal parameter defined in § 3 and σ is the non-dimensional frequency; these parameters are the same as those used in § 5 of HM. For low frequencies ($\sigma \gtrsim 1$) (7.8*a*) reveals that τ is large, as expected (see also figure 3).

Note first of all from figure 3 that, for each value of the non-dimensional correlation scale α, the e-folding distance rapidly attains a minimum value as σ increases away from unity. As α decreases (corresponding to rougher coastlines) these minima occur at progressively higher frequencies (shorter wavelengths), and we conclude that the most rapidly decaying K-waves (on a scale of wavelength) have wavelengths of the same order as the correlation scale, i.e. satisfy $\alpha(\sigma^2-1)^{\frac{1}{2}} = O(1)$. At very high frequencies, figure 3 corroborates our earlier asymptotic analysis, the linear form of the curves for large σ being in qualitative agreement with our conclusion

$$\tau \sim l^2/f\langle\xi^2\rangle,$$

or equivalently
$$\epsilon^2\omega\tau \sim \sigma\alpha^2.$$

For fixed values of σ an examination of the linear portions of the curves also confirms that the e-folding distance increases approximately quadratically with the correlation scale α. At very small values of α (rough coasts) the e-folding distance increases very slowly with increasing frequency, indicating that all short waves are rapidly attenuated.

In the absence of an incident Poincaré wave ($P^* \equiv 0$), equation (6.12) for the directivity $Q(\theta)$ of the scattered P-wave noise reduces to

$$Q(\theta)\left\{1+\frac{\pi f}{\omega b\cos\theta}\left(\frac{\omega}{s}\right)^3(1+b\sin\theta)^2\,\Phi\left[\frac{\omega}{s}(1+b\sin\theta)\right]\right\}$$
$$= K(y,t)\,2fb^2\left(\frac{\omega}{s}\right)^3\cos^2\theta\left(\frac{1+b\sin\theta}{1-b\sin\theta}\right)\Phi\left[\frac{\omega}{s}(1+b\sin\theta)\right], \qquad (7.9)$$

in which $O(\xi^4)$ terms describing multiple interactions among the P-waves have been discarded. When $K(y,t)$ has the form given in (7.3) it is clear that (7.9) determines the distribution of P-waves radiating from a moving source at the current location $y = -st$ of the Kelvin wave.

Let us examine the nature of the angular distribution $Q(\theta)$ in the limit of large correlation scale $c^2 = (\omega l/2s)^2 \gg 1$. The second term in the curly brackets on the left of (7.9) is important only at grazing scattering angles ($\theta \sim \pm\frac{1}{2}\pi$); we shall neglect its contribution and examine the validity of the approximation *a posteriori*.

Thus for $c^2 \gg 1$ we have from (7.9)

$$Q(\theta) \simeq \frac{K(y,t)\,2\omega f(\omega^2-f^2)(1-b)\,\Phi_0\exp\{-c^2(1-b)^2\}}{s^2b(1+b)}$$
$$\times(\theta+\tfrac{1}{2}\pi)^2\exp\{-(\theta+\tfrac{1}{2}\pi)^2c^2b(1-b)\}. \qquad (7.10)$$

This expression has its maximum value at

$$\theta_m+\tfrac{1}{2}\pi \equiv \Delta\theta = 1/c\,[b(1-b)]^{\frac{1}{2}} \ll 1. \qquad (7.11)$$

Accordingly the θ_m direction makes a small angle $\Delta\theta$ with the direction ($\theta = -\frac{1}{2}\pi$) of propagation of the Kelvin wave along the coast. In other words the energy of the K-wave is predominantly scattered in the forward direction into a narrow Gaussian beam of Poincaré waves of width $\sim\Delta\theta$ which makes a small angle $\Delta\theta$ with the direction of propagation of the K-wave.

The neglect of the second term in the curly brackets on the left of (7.9) can now be seen to require that

$$\frac{\pi\omega^2f(1+b\sin\theta_m)^2\,\Phi_0}{s^2b}\exp\{-c^2(1+b\sin\theta_m)^2\} \ll \cos\theta_m, \qquad (7.12)$$

i.e. that
$$\frac{8\pi c^3(1-b)^2\,\Phi_0\exp\{-c^2(1-b)^2\}}{\sigma b l^3} \ll \Delta\theta,$$

or
$$\frac{\langle\xi^2\rangle}{l^2} \ll \frac{\sigma}{4c^2(1-b)^2}\left(\frac{b}{\pi(1-b)}\right)^{\frac{1}{2}}\exp\{c^2(1-b)^2\}. \qquad (7.13)$$

Taking $c = 10$ and $b = 0{\cdot}9$ ($\sigma \simeq 2{\cdot}3$), say, and using (7.5) and (7.11), this condition implies that

$$\langle\xi^2\rangle^{\frac{1}{2}}/l \ll 0{\cdot}16, \qquad (7.14)$$

i.e. that the correlation scale must also be large compared with ξ.

A similar analysis justifies the neglect in the above discussion of the third term on the right of (6.12), which describes multiple scattering of the P-wave noise (cf. Howe 1974*a*, § 5).

8. The generation of a Kelvin wave by a normally incident Poincaré wave

Suppose next that a Poincaré wave is incident normally along the whole of the coast, the wave front arriving at time $t = 0$, and that

$$K(y, 0) = 0. \tag{8.1}$$

When variations with y of the incident P-wave can be neglected the leading approximation to (6.1) becomes

$$\partial K/\partial t + K/\tau = S_K H(t). \tag{8.2}$$

This problem is a simple model of an extensive storm surge that originates far out at sea and propagates towards the coast.

The solution

$$K = S_K \tau\{1 - e^{-t/\tau}\} \tag{8.3}$$

may be expressed in the form

$$\frac{K/\tau}{P*} = bC(\theta_0 = 0)(1 - e^{-t/\tau}), \tag{8.4}$$

where the coefficient C is defined in (3.12) and (for later convenience) is given here as a function of the angle of incidence θ_0:

$$C(\theta_0) = \frac{4\pi\omega^2 f(1 + b\sin\theta_0)}{s^3(1 - b\sin\theta_0)}\,\Phi\left[\frac{\omega}{s}(1 + b\sin\theta_0)\right]. \tag{8.5}$$

The quantity $(K/\tau)/P^*$ represents the ratio

$$E_K(t) \equiv \frac{\text{flux of energy into } K\text{-wave}}{\text{incident energy flux}}.$$

When the steady state is attained ($t \to \infty$), $E_K(t) \to bC \equiv E_K$, which was designated the 'K-wave efficiency factor' in HM (see figure 2 of HM for the variation of E_K as a function of the angle of incidence in the case of a Gaussian coastal spectrum). For finite times t and large values of the relaxation time τ (corresponding to the low frequency case $\omega/f \gtrsim 1$, $b \ll 1$), (8.4) shows that $K(t) \simeq bC(0)\,P^* t$. Since b is small this implies that the energy content of the Kelvin wave increases very slowly. The dependence of $\log_{10} E_K(t)$ on the frequency ω of the incident Poincaré wave at various times t and for three different values of the correlation scale $\alpha = l/2r$ is illustrated in figure 4. The dimensionless parameters are the same as those used previously (§ 7). The time $T = tf$ is measured in units of the inverse Coriolis parameter; at a latitude of 45° say, $T = 1$ corresponds to about 2·8 h.

Observe that at a fixed frequency the scattering becomes considerably more efficient as the correlation scale α decreases, the total change from figure 4(a) to figure 4(c) being typically two or more orders of magnitude. We also note that, for fixed T and α, $\log_{10} E_K(t)$ rapidly increases with σ until a maximum is reached. That is, K-waves are most efficiently generated at all times when the correlation scale is

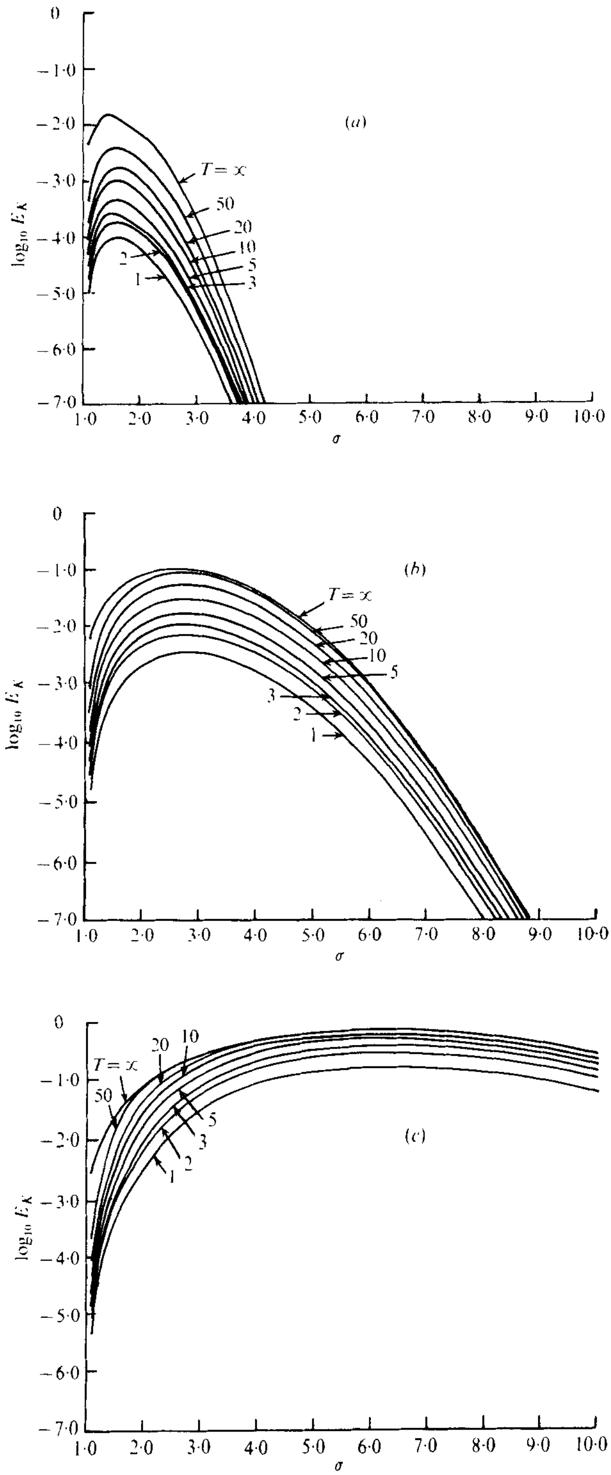

FIGURE 4. $\mathrm{Log}_{10}\, E_K(t)$ *vs.* frequency $\sigma = \omega/f$ for a normally incident P-wave. $E_K(t)$ is given by the right-hand side of (8.4), in which the Gaussian spectrum (7.5) is used. (*a*) $\alpha = 1\cdot0$, (*b*) $\alpha = 0\cdot5$, (*c*) $\alpha = 0\cdot2$. $T = tf$ is a non-dimensional time and in all cases $\epsilon = 0\cdot1$.

comparable to the wavelength. But this is precisely when the e-folding distance of a K-wave is greatest (see figure 3). Thus it appears that the two processes – generation and dissipation due to coastal irregularities – are likely to be very much in balance.

9. Localized forcing of Kelvin waves

A Poincaré wave incident over a finite portion of the coast may be taken to model a localized storm surge. The problem now is to determine the Kelvin-wave response at remote positions along the coast. If the storm is centred about $y = 0$ on the coast and extends over a region which is small compared with the dissipation scale $s\tau$, the spatial distribution of power in the incident wave may be represented by

$$P^* = P_0^* \delta(y),$$

where P_0^* is the total incident power.

If the Poincaré wave arrives at time $t = 0$, the leading approximation to the Kelvin-wave equation (6.1) becomes

$$\left(\frac{\partial}{\partial t} - s\frac{\partial}{\partial y}\right) K + \frac{K}{\tau} = bP_0^* C(\theta_0)\,\delta(y)\,H(t), \tag{9.1}$$

where $C(\theta_0)$ is given by (8.5).

Introduce the Green's function $G(y,t)$ which is the causal solution of

$$\left(\frac{\partial}{\partial t} - s\frac{\partial}{\partial y}\right) G + \frac{G}{\tau} = \delta(y)\,\delta(t), \tag{9.2}$$

viz.

$$G(y,t) = \delta(y+st)\,H(t)\,e^{-t/\tau}. \tag{9.3}$$

The solution of (9.1) may now be expressed as the convolution product

$$K(y,t) = \iint_{-\infty}^{\infty} bP_0^* C(\theta_0)\,\delta(Y)\,H(T)\,G(y-Y, t-T)\,dY\,dT,$$

i.e.

$$K(y,t) = bP_0^* C(\theta_0) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(Y)\,H(T)\,\delta[y-Y+s(t-T)]\,H(t-T)\,e^{-(t-T)/\tau}\,dY\,dT$$

$$= bP_0^* C(\theta_0) \int_0^t \delta[y+s(t-T)]\,e^{-(t-T)/\tau}\,dT. \tag{9.4}$$

The argument of the δ-function vanishes at $T = t + y/s$, which lies within the range of integration provided that

$$0 < y/s + t < t, \tag{9.5}$$

i.e. for $y < 0$. Hence, for $t > 0$,

$$K(y,t) = H(-y)\,H\left(t+\frac{y}{s}\right)\frac{b}{s}P_0^* C(\theta_0)\,e^{y/s\tau}. \tag{9.6}$$

This result reveals that, at time t after the arrival of the storm surge at the coast, the Kelvin wave has swept out an exponentially decaying envelope which stretches a distance st along the coast from the storm centre. Note in particular that the intensity of the Kelvin wave saturates immediately the wave front $y = -st$ arrives. The profile of the envelope is time independent, so that ultimately (as $t \to \infty$) the wave amplitude becomes an exponentially decaying function of y (<0) alone. When this steady state is attained (and in the absence of other dissipative mechanisms) all of the power incident on the coast from the Poincaré wave is re-radiated as diffuse P-waves. The distance D over which the intensity of the Kelvin wave is reduced to 10% of its magnitude in the vicinity of the storm is given by

$$1-e^{-D/s\tau} = 0{\cdot}9,$$

i.e. $D \simeq 2{\cdot}3\, s\tau$. Numerical estimates for D can be computed from figure 3 when the frequency ω of the storm surge is specified. As an illustration consider the case $\alpha = l/2r = 0{\cdot}5$ and suppose that $\langle\xi^2\rangle^{\frac{1}{2}}/l = 0{\cdot}1$. Let the radian frequency of the incident wave be given by $\omega = 3f$, corresponding to a characteristic storm surge period of about 6 h at a latitude of 45°. These figures imply that $\epsilon = \langle\xi^2\rangle^{\frac{1}{2}}(s/f)^{-1} = 0{\cdot}1$, and it follows from figure 3 that $\omega\tau \simeq 13$. Hence the Kelvin-wave relaxation distance

$$D \simeq 2{\cdot}3\, s\tau = 2{\cdot}3\,\frac{(\omega\tau)\,(s/f)}{\omega/f} \simeq 10l.$$

Since $l = r = O(10^3\,\text{km})$ this distance is unlikely to be relevant in practice because of the intervention of more severe coastal irregularities.

Consider next the response produced by a localized storm surge which is moving parallel to the coast. Much of the early literature on storm surges is concerned with such problems (see, for example, Thomson 1970), but little attention has been paid to the effects of coastal irregularities. We now have

$$\left(\frac{\partial}{\partial t} - s\frac{\partial}{\partial y}\right)K + \frac{K}{\tau} = S_0\,\delta(y-Vt), \tag{9.7}$$

where the right-hand side is a formal representation of the storm, S_0 being a measure of its intensity. Using the Green's function (9.3) we find

$$\begin{aligned} K(y,t) &= S_0 \iint_{-\infty}^{\infty} \delta(Y-VT)\,\delta[y-Y+s(t-T)]\,H(t-T)\,e^{-(t-T)/\tau}\,dY\,dT \\ &= S_0 \int_{-\infty}^{t} \delta\{y+st-T(V+s)\}\,e^{-(t-T)/\tau}\,dT. \end{aligned} \tag{9.8}$$

We examine first of all the 'resonant' case in which $V = -s$. This apparently singular situation should be interpreted as follows. An arbitrary storm may be Fourier decomposed into component, harmonic storm waves; the case $V = -s$ therefore applies to that constituent group of storm waves whose group velocity parallel to the coast is just equal to the propagation velocity of Kelvin waves. The solution (9.8) is then

$$K = S_0\,\tau\,\delta(y+st). \tag{9.9}$$

Since $\tau = O(1/\epsilon^2) \gg 1$ this describes a steady Kelvin wave of relatively large amplitude and, as is usual for a harmonic oscillator driven at resonance, the intensity of the K-wave varies inversely with the damping coefficient $1/\tau$. The importance of this case is that it leads to the most rapid transfer of energy to the Kelvin wave. Thus the strong coupling with the storm at resonance could, in principle, lead to its very rapid dissipation.

When $V \neq -s$, we have

$$K(y,t) = \frac{S_0}{|V+s|} H\left(\frac{Vt-y}{V+s}\right) \exp\left\{\frac{-(Vt-y)}{\tau(V+s)}\right\}, \tag{9.10}$$

a result which shows that the K-wave amplitude always decays with distance from the storm centre $y = Vt$. For subcritical storm propagation velocities $|V| < s$, the Kelvin waves propagate into the wake or ahead of the storm according as $V \gtrless 0$. At supercritical velocities the Kelvin waves propagate away from the storm and into the wake for $V > s$; when $V < -s < 0$, the Kelvin waves propagate in the wake but in the same direction as the storm.

10. Multiply scattered Poincaré waves

In this section we examine the effect that multiple scattering has on the distribution $Q(\theta)$ of the P-wave noise. In the single-scattering approximation (6.12) reduces to

$$Q(\theta) \equiv Q_S(\theta) = K_I + P_I, \tag{10.1}$$

where the terms on the right-hand side denote respectively the first and last components on the right of (6.12), i.e. the energy fluxes into the P-wave noise from the Kelvin wave and the incident Poincaré wave. These terms are generally $O(\epsilon^2)$ relative to the incident wave.

When multiple-scattering terms are retained in (6.12) we can write

$$Q(\theta) \equiv Q_m(\theta) = \frac{K_I + P_I}{1 + [A(\theta) + B(\theta)] \sec\theta}, \tag{10.2}$$

where

$$\left.\begin{aligned} A(\theta) &= \frac{\pi\omega^2 f(1+b\sin\theta)^2}{s^3 b} \Phi\left[\frac{\omega}{s}(1+b\sin\theta)\right] > 0, \\ B(\theta) &= \frac{\cos\theta\,\omega^3 b^3}{s^3} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \Phi\left[\frac{\omega}{s} b(\sin\theta - \sin\phi)\right] F(\phi,\theta)\,d\phi > 0. \end{aligned}\right\} \tag{10.3}$$

Equation (10.2) follows from (6.12) when the $O(\epsilon^4)$ input of energy into the P-waves radiating in the θ direction from other P-waves radiating in the ϕ direction

$$(-\tfrac{1}{2}\pi < \phi < \tfrac{1}{2}\pi),$$

which corresponds to the first term in the curly brackets of the integrand on the right of (6.12), is neglected.

The coefficients $A(\theta)$ and $B(\theta)$ are $O(\epsilon^2)$ and account respectively for the loss of energy from P-waves radiating in the θ direction into the Kelvin wave and into P-waves radiating in the ϕ direction. Equations (10.3) and (6.13) show that A and B remain finite as $\theta \to \pm\frac{1}{2}\pi$, so that provided that $\sec\theta = O(1)$, i.e. away from near-grazing scattering angles, $Q_S/Q_m = 1 + O(\epsilon^2)$ and multiple scattering is not significant. For angles close to $\pm\frac{1}{2}\pi$ however, the term in $\sec\theta$ in (10.2) becomes important, so that $Q_S/Q_m = O(\sec\theta)$, i.e. at these angles multiple scattering reduces the intensity of the P-wave radiation into the open ocean.

The behaviour of the power ratio Q_S/Q_m is illustrated in figure 5. The quantity

$$\begin{aligned} \Delta &= 10\log_{10}(Q_S/Q_m) \\ &= 10\log_{10}\{1 + [A(\theta)+B(\theta)]\sec\theta\}, \end{aligned} \tag{10.4}$$

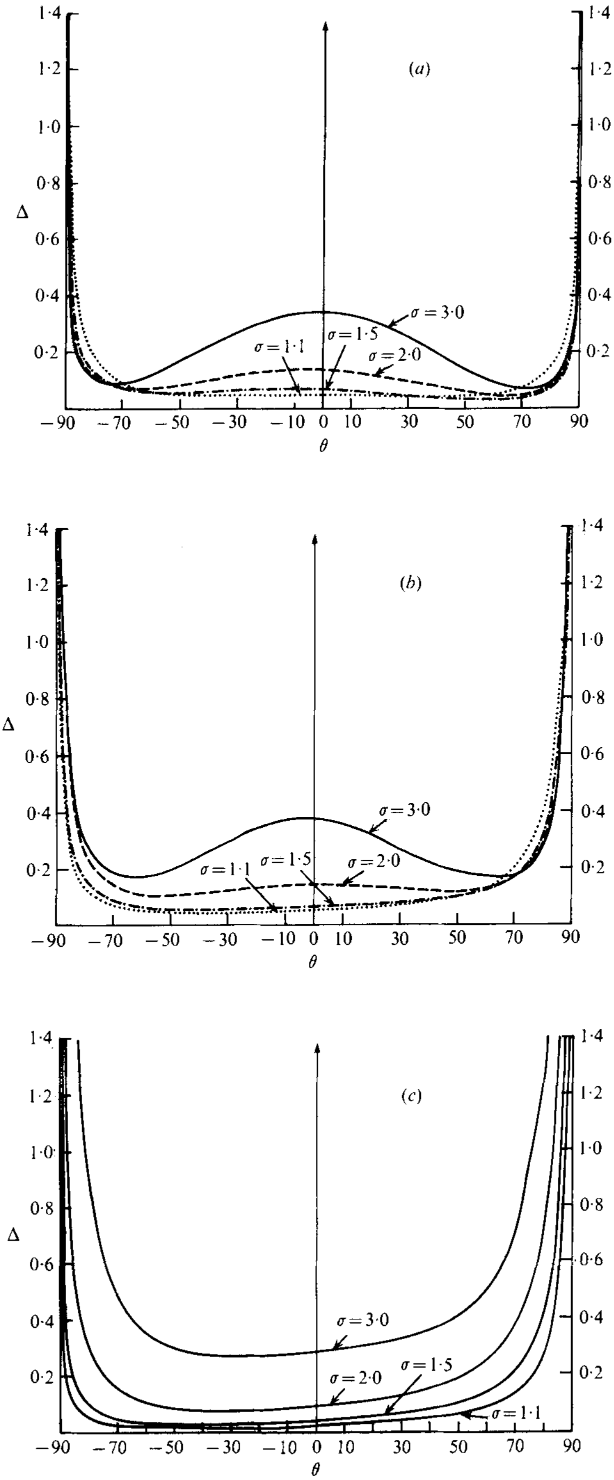

FIGURE 5. Plots of the power ratio Δ *vs.* θ as given by (10.4) for the case of the Gaussian spectrum (7.5). In all cases $\epsilon = 0{\cdot}1$. (*a*) $\alpha = 1{\cdot}0$, (*b*) $\alpha = 0{\cdot}5$, (*c*) $\alpha = 0{\cdot}2$.

which is always positive because $Q_m < Q_S$, is plotted as a function of θ for $\epsilon = 0{\cdot}1$ and different values of the frequency ω/f and the correlation scale α.

Figure 5(a) depicts the case of a 'smooth' coast with $\alpha \equiv l/2r = 1$. As $\theta \rightarrow \pm\frac{1}{2}\pi$, Δ becomes large, but in addition there is a local maximum near $\theta = 0$ which is more pronounced at the higher frequencies. Thus at high frequencies multiple scattering is marginally significant near normal scattering angles, a result which is directly attributable to the effect of rotation (i.e. the presence of the Coriolis parameter f) in the function $F(\phi,\theta)$ appearing in the definition of $B(\theta)$. Similar remarks apply to figure 5(b), for which $\alpha = 0{\cdot}5$. However a slight skewness is now observable at the lower frequencies, Δ being somewhat larger near $\theta = +\frac{1}{2}\pi$ than near $\theta = -\frac{1}{2}\pi$. This effect can be traced to the behaviour of the Kelvin-wave coefficient $A(\theta)$ in (10.4):

$$A(\theta \simeq = \tfrac{1}{2}\pi) > A(\theta \simeq -\tfrac{1}{2}\pi),$$

indicating that, relative to the direction of propagation of the K-wave, more energy is fed into the Kelvin wave from the P-wave noise at back-scattering angles. In the case of a 'rough coast' (figure 5c), for which $\alpha = 0{\cdot}2$, the preference for multiple scattering at near-normal angles has disappeared, but the skewness at low frequencies is more significant.

Of course the logarithmic scale used in these plots implies that, except at angles very close to grazing and at relatively high frequencies, the actual numerical values of the power ratio are not significantly different from unity.

11. Conclusion

In this paper we have developed a multiple-scattering theory which describes the long distance/time interaction of waves on a uniformly rotating sheet of homogeneous fluid with a coastline which is straight except for small random deviations. The theory is expected to be valid provided that the characteristic wavelength is large compared with the root-mean-square coastal irregularity, and has been applied to several idealized models of tidal and storm surges.

In particular, emphasis was placed on the determination of the distribution of scattered waves produced by a localized or extensive Poincaré wave incident on the coast. Our conclusions relating to the generation of Kelvin waves invite a comparison with other possible forcing mechanisms, such as the effect of atmospheric pressure fluctuations, surface wind stress and earth tides. Further, it would be of great interest to compare the degradation of a Kelvin wave by coastal irregularities with that caused, for example, by bottom friction. These important issues have not been examined here, and their investigation would require a careful analytical modelling of the corresponding energy sources and sinks in the general kinetic equation. Such an approach is now the object of modern studies in ocean wave dynamics (cf. the recent study of internal waves in the deep ocean by Müller & Olbers 1975), and it would appear to be appropriate to attempt a similar study for the dynamics of Kelvin waves.

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